Hierarchic trees with branching number close to one: Noiseless Kardar-Parisi-Zhang equation with additional linear term for imitating two-dimensional and three-dimensional phase transitions

D. B. Saakian
Yerevan Physics Institute, Alikhanian Brothers Street 2, Yerevan 375036, Armenia
(Received 25 October 2001; published 28 June 2002)

An imitation of two-dimensional (2D) field theory is formulated by means of a model on the hierarchic tree (with branching number close to one) with the same potential and the free correlators identical to 2D correlator ones. Such a model carries on some features of the original model for certain scale invariant theories. The renormalization group equation for the free energy is the noiseless Kardar-Parisi-Zhang equation with an additional linear term.

DOI: 10.1103/PhysRevE.65.067104 PACS number(s): 64.60.Ht, 05.45.−a, 05.70.Ln

Usually one understands the universality of phase transition considering different Hamiltonians in the same space.

We propose to follow another path: by keeping the Hamiltonian fixed to simplify the space geometry as much as possible retaining two point correlators and three point (for isosceles triangles) correlators.

There is a precedent of such a situation (a model in simplified space holds some features of a model in ordinary space). In his paper [1], Baxter observed that the equations of free energy in the two-dimensional (2D) Ising model with anisotropic couplings are similar to those in the Bethe lattice (while critical indices are different). In this work we are going to look for a similar situation for field theoretical models.

If the action of original theory consists of the Laplacian and a potential, our model experiences the space dimension through the behavior of the Green function

$$G(x,x')\sim\frac{1}{r(x,x')^{d-2}},\quad d\neq 2\quad \text{and}$$

$$G(x,x')\sim\ln\frac{1}{r(x,x')},\quad d=2. \tag{1}$$

The total volume is

$$\left(\frac{L}{a}\right)^{d}, \tag{2}$$

where L and a are the infrared and ultraviolet cutoffs, $r(x,x')$ is the distance. The Euclidean geometry contains too much construction. One can rotate a point around some center and write out a close circle. Let us now consider some metric space with the following properties: (a) For every pair of points there is a distance $r(x,x')$. (b) We have some measure at every point $d\mu_s(x)$ with the total measure $\int d\mu_s = R^d$. (c) One can construct a quadratic form with corresponding asymptotics (1) for the Green function.

We are going to construct statistical mechanics models on the simplest space, which supports points (a)–(c). We hope that due to the universality these models will acquire some properties of models in d-dimensional space. To realize this program we will use certain ideas from the theory of the random energy model (REM) [2–4]. In Ref. [5] a relation of the 2D quantum Liouville model to the REM and to the directed polymer on the Cayley tree was established.

Our present analysis shows that the connection with the REM is not a specific feature of the Liouville model and works well also for other conformal models. Moreover, using similar ideas we intend to construct general 2D quantum models in the ultrametric space and thereby generalize the above-mentioned connection between the quantum field theoretical models and those defined on the hierarchical lattices.

Let us consider a hierarchic tree with the branching number q. We begin with integer q, then continue the obtained expressions analytically to the point $q\to 1$. Instead of d-d Euclidean space now we have q^K end points, where K is a number of hierarchic levels. First we define the fields f_l on branches of a tree. The field ϕ at the end point x is defined as

$$\phi(x)=f_0+\sum_l f_l. \tag{3}$$

The summation in Eq. (3) is made along the trajectory of point x, connecting it with the origin of the tree. We define v at the hierarchy level j as

$$v=\frac{jV}{K}. \tag{4}$$

Now determine the kinematic part of the action for the field $\phi(x)$

$$-\sum \frac{K}{2V}f(v,l)^2. \tag{5}$$

Then the partition under the potential $U(\phi)$ is

$$\int df\exp\left\{-\sum_{v=v_1}^{v_K\equiv V}\frac{K}{2V}f(v,l)^2\right\}\exp\left\{-\sum_x U(\phi(x))\right\}. \tag{6}$$

We have for the correlator

$$\langle\phi(x)\phi(x')\rangle=V-v. \tag{7}$$

For usual 2D models with

$$\int d\phi_0 d\phi \exp\left\{-\frac{1}{8\pi}dx^2\nabla\phi(x)^2\right\}\exp\left\{-\int dxU(\phi(x))\right\}, \tag{8}$$

the total surface area is equal to R^2, and the correlators read

$$\langle\phi(x)\phi(x')\rangle=\ln\frac{L^2}{r^2}. \tag{9}$$

It is possible to take n-component fields in Eq. (8) instead of the one-component field $\phi(x)$.

We can determine the distance from the equality $v=\ln\, r^2$. Then our correlator (7) coincides with the 2D one (9).

What is the advantage of representation (6)? We are in a position to calculate the partition function by means of iterations. This is well known for models on hierarchical lattices [1]. Let us take some large number K and derive

$$I_1(x)=\sqrt{\frac{K}{2V\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{K}{2V}y^2-U(x+y)\right\}dy,$$

$$I_{i+1}(x)=\sqrt{\frac{K}{2V\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{K}{2V}y^2\right\}[I_i(x+y)]^q dy,$$

$$Z=\lim_{K\to\infty}[I_K(0)]^q. \tag{10}$$

As for the determination of the partition function, we need only Eq. (10) and we can define our model for any value of q consideration of the analytical continuation of Eq. (10). Let us consider the limit

$$q\to 1,\ K\to\infty \quad q^K=e^V. \tag{11}$$

Using the small factor $(q-1)$, it is possible to introduce continuous measures $d\mu_x, d\mu_l$, construct perturbative field theory on this ultrametric space, and calculate diagrams. In reality we need expressions for the propogator (7), as well as the total volume measure inside the sphere with maximal hierarchic distance v given by the equality

$$\int d\mu_x=e^v-1. \tag{12}$$

For finite or large values of q considered in [3] and [4] it is impossible (or too difficult) to define the perturbative regime.

Let us consider carefully Eq. (10) at the limit (11). We introduce a variable $w(v,x)=I_{Kv/V}(x)$ and consider the limit $V/K\ll 1$. For the differential dv we have the expression V/K. Let us also take

$$q-1=\frac{V}{K}\equiv dv. \tag{13}$$

Using expression $x^q\approx x[1+\ln x(q-1)]$ it is easy to obtain

$$\frac{dw}{dv}=w\ln w+\frac{1}{2}\Delta w. \tag{14}$$

After the replacement $w=\exp[-u(t,x)]$ we arrive at

$$\frac{du}{dv}=\frac{1}{2}\Delta u-\frac{1}{2}(\nabla u)^2+u,$$

$$x<-\infty x<\infty, \quad 0\leqslant v\leqslant V, \quad u(0,x)=U(x), \tag{15}$$

where $U(x)$ is a potential in Eq. (8). The dimension n of the space where this equation is formulated is equal to the number of different fields $\phi(x)$ in Eq. (8). Having an expression for $u(v,x)$ we obtain for the free energy

$$\ln Z=-u(V,0). \tag{16}$$

For the free energy $u(v,x)$ we have the noiseless Kardar-Parisi-Zhang equation (15) with an additional linear term. It is easy to find several solutions [Eqs. (14) and (15)].

When there is no dependence from x,

$$u(v,x)=\text{const}\exp(v). \tag{17}$$

If at the boundary $v=0$ the potential $U(x)=x^2$, then it holds such a form for any value of v,

$$u(v,x)=\phi_0(v)+x^2\phi_1(v). \tag{18}$$

For the case of discrete models (like Ising or Potts) on a hierarchic lattice the stable points of iteration equations give the solution of the problem. In our case it has a narrower meaning [there is another interesting situation, when $\sim 1/\exp(V)$]:

First the static solution is

$$w(v,x)=1, \quad u(v,x)=0. \tag{19}$$

The next one is

$$w(v,x)=0, \quad u(v,x)=\infty. \tag{20}$$

It is possible to find other real solutions. We have an even solution $u(x)=u(-x)$, where the potential is finite only in the finite interval,

$$u=\infty, \quad -\infty<x<-\int_{z_0}^{\infty}\frac{dz}{\sqrt{1+2z+c\exp(2z)}},$$

$$1+2z_0+c\exp(2z_0)=0. \tag{21}$$

Outside that interval there is a nontrivial dependence between u and x,

$$x=-\int_u^{\infty}\frac{dz}{\sqrt{1+2z+c\exp(2z)}},$$

$$\infty>u>z_0$$

$$0>x>-\int_{z_0}^{\infty}\frac{dz}{\sqrt{1+2z+c\exp(2z)}}. \tag{22}$$

Here c is some positive constant.

Let us use the same approach for the case of $d>2$. For the volume in d-d space one has $\sim a^d$. If we identify it with our q^L, then we derive $a=q^{(1/d)L}$. The fields f_l are defined on the branches of the tree, f_0, f_1 at the origin. Let us define the free field action:

$$\phi(x)=f_0+f_1+\sum_{vl} f(v,l). \tag{23}$$

The summation in Eq. (24) is along the trajectory of point X. Now determine the kinematic part of the action for the field $\phi(x)$,

$$A=\tfrac{1}{2}\left[f_1^2+\sum_{vl} \exp(-\alpha v)f(v,l)^2/\alpha\right]. \tag{24}$$

If one takes $\alpha=(d-2)/d$ for the combined field, then

$$\langle\phi(x)\phi(x')\rangle=\exp(\alpha v)\sim\frac{L^{(d-2)}}{r(x,x')}, \tag{25}$$

where L is the maximal distance in the model (the infrared cutoff). We should solve the equation like Eq. (15):

$$\frac{du}{dv}=\frac{1}{2}\alpha\exp[\alpha v]\Delta u-\frac{1}{2}(\boldsymbol{\nabla}u)^2+u,$$

$$u(0,x)=U(x). \tag{26}$$

We have given a simplified, approximate method for the 2D field theoretical models and derived a different version of the Kolmogorov equation. We hope that the bulk structure, the two and three point correlators (for isosceles triangles) in our approach, are the same as in 2D models.

It is possible to check the equivalence of models on our trees with some segment of d-d field theory by means of direct numerical calculation of Eqs. (14) and (27) for a field version of the 3D Ising model.

I am grateful to ISTC, Grant No. A-102, for partial financial support and B. Derrida, W. Janke, L. Kadanoff, C. Lang, V. Volpert, Y. Sinai, J. Peinke, and R. Gulghazaryan for discussions.

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